7. Strong Markov property. Stopping times are amazingly useful for studying Brownian motion. There are two key properties of BM that matter. The strong Markov property is one, the other is optional sampling theorem as applied to various martingales associated to BM. In this problem set, just use the strong Markov property without invoking martingales.

33. Let $X_t = W_t + t^{\alpha}$. Then 0 is almost surely an isolated zero of X for $\alpha < \frac{1}{2}$ and is almost surely not an isolated zero for $\alpha > \frac{1}{2}$. [*Optional:* Can you also figure out what happens for $\alpha = \frac{1}{2}$?]

34. For $A \subseteq \mathbb{R}$, let τ_A be the hitting time of the set A by standard 1-dimensional BM. Let $a < 0 < b$.

1. (One sided hitting time). Show that $\mathbf{P}(\tau_a < \infty) = 1$ but $\mathbf{E}[\tau_a] = \infty$.
2. (Two sided hitting time). Show that $\mathbf{E}[\tau_{a,b}] < \infty$, in fact, the random variable $\tau_{a,b}$ has sub-Gaussian tails, that is, $\mathbf{P}(\tau_{a,b} > t) \le e^{-\gamma t^2}$ for some $\gamma > 0$ that depends on a, b.

[*To think:* For planar Brownian motion, which sets have $\mathbf{P}_0(\tau_A < \infty) = 1$? Which sets have $\mathbf{E}_0[\tau_A] < \infty$? The general questions are hard, but take special sets and try to figure the answer.]

35. Let $\mathbf{W} = X + iY$ be planar Brownian motion started at i. Let $\tau = \inf\{t : \mathbf{W}_t \in \mathbb{R}\}$ be the hitting time of the real line.

1. Show that X_τ has Cauchy distribution with density $\frac{\alpha}{\alpha^2+t^2}\frac{dt}{\pi}$ for some $\alpha > 0$. [**Hint:** Scaling relations plus strong Markov property].
2. Show that $\alpha = 1$. [**Hint:** Consider also τ', the hitting time of the line $x = -1$].
3. Generalize to the question of the distribution of d-dimensional Brownian motion when it hits a $(d-1)$-dimensional hyperplane.

36. Let W be 1-dimensional Brownian motion and let M be its running maximum. Fix $t > 0$ in this problem.

1. For $a > 0$, $x > 0$, use the reflection principle to show that $\mathbf{P}(M_t > a, W_t < a - x) = \mathbf{P}(W_t > a + x)$.
2. Use this to compute the joint density of (M_t, W_t).
3. Compute the density of $M_t - W_t$ and check that it is the same as the density of $|W_t|$.

[**Note:** The last part, together with the fact that $M - W$ and $|W|$ are Markov processes, implies Lévy's identity].